Finite element formulations at finite strains

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1 List of equations

The total strain is

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}), \tag{1}$$

The first Piola-Kirchhoff stress $\boldsymbol{P}$ and the second Piola-Kirchhoff stress $\boldsymbol{S}$ are given by

$$\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}, \tag{2}$$

and

$$\boldsymbol{S} = \frac{\partial \psi}{\partial \boldsymbol{E}}. \tag{3}$$

respectively, where ψ is the free energy density. The Cauchy stress is

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T, \tag{4}$$

where $J = \det\boldsymbol{F}$. The Kirchhoff stress is defined as $\boldsymbol{\tau} = J\boldsymbol{\sigma}$.

The equilibrium equation in the reference configuration is given by

$$\nabla_0 \cdot \boldsymbol{P} = \boldsymbol{0}, \tag{5}$$

where ∇_0 represents the gradient operator with respect to the reference configuration.

2 Weak form and linearization

The finite element scheme has been derived mostly by following Chapter 10 of [1]. We write the weak form of the balance relation (5) as

$$\mathcal{R} = -\int_{\Omega_0} (\nabla_0 \cdot \boldsymbol{P}) \cdot \delta\boldsymbol{u}\, dV_0 = 0, . \tag{6}$$

where $\delta\boldsymbol{u}$ is the virtual displacement. Using the Gauss divergence theorem and considering the traction boundary condition we rewrite $\mathcal{R}$ in (6) as

$$\begin{aligned} \mathcal{R} &= \int_{\Omega_0} \boldsymbol{P} : \nabla_0(\delta\boldsymbol{u})\, dV_0 - \int_{\Omega_0} \nabla_0 \cdot (\boldsymbol{P}^T \cdot \delta\boldsymbol{u})\, dV_0 \\ &= \int_{\Omega_0} \boldsymbol{P} : \nabla_0(\delta\boldsymbol{u})\, dV_0 - \int_{\partial\Omega_0} \boldsymbol{p}_{\text{ex}} \cdot \delta\boldsymbol{u}\, dA_0, \end{aligned} \tag{7}$$

where $\boldsymbol{p}_{\text{ex}}$ is the external traction on the external boundary of the body $\partial\Omega_0$. Using the relation $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$ we can rewrite (7) as

$$\mathcal{R} = \int_{\Omega_0} \boldsymbol{S} : \boldsymbol{F}^T \nabla(\delta \boldsymbol{u}) \boldsymbol{F} \, dV_0 - \int_{\partial\Omega_0} \boldsymbol{p}_{\text{ex}} \cdot \delta \boldsymbol{u} \, dA_0, \tag{8}$$

where $\nabla(\cdot)$ represents the gradient operator with respect to the deformed configuration.

We will now work on the first integrand of (8). Noticing that $E_{IJ} = \frac{1}{2}(F_{iI}F_{jJ} - \delta_{IJ})$ we calculate

$$\begin{aligned} \delta E_{IJ} &= \frac{1}{2}\frac{\partial \delta u_i}{\partial X_I} F_{iJ} + \frac{1}{2}\frac{\partial \delta u_i}{\partial X_J} F_{iI} \\ &= \frac{1}{2}\frac{\partial \delta u_i}{\partial x_k} F_{kI} F_{iJ} + \frac{1}{2}\frac{\partial \delta u_i}{\partial x_k} F_{kJ} F_{iI} \\ &= \frac{1}{2}\frac{\partial \delta u_k}{\partial x_i} F_{iI} F_{kJ} + \frac{1}{2}\frac{\partial \delta u_i}{\partial x_k} F_{kJ} F_{iI} \\ &= \delta\epsilon_{ik} F_{kJ} F_{iI}, \end{aligned} \tag{9}$$

where $\delta\epsilon_{ik} = \dfrac{1}{2}\left(\dfrac{\partial \delta u_k}{\partial x_i} + \dfrac{\partial \delta u_i}{\partial x_k}\right)$. Noticing that $\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T$ is symmetric and using (9) we see that $\boldsymbol{S} : \boldsymbol{F}^T \nabla(\delta \boldsymbol{u}) \boldsymbol{F} = \boldsymbol{S} : \delta \boldsymbol{E}$. Hence using $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T}$ the weak form (6) can be finally rewritten as

$$\mathcal{R} = \int_{\Omega_0} \tau_{ij}\delta\epsilon_{ij} \, dV_0 - \int_{\partial\Omega_0} \boldsymbol{p}_{\text{ex}} \cdot \delta \boldsymbol{u} \, dA_0 = 0, \tag{10}$$

or equivalently,

$$\mathcal{R} = \int_{\Omega_0} S_{IJ}\delta E_{IJ} \, dV_0 - \int_{\partial\Omega_0} \boldsymbol{p}_{\text{ex}} \cdot \delta \boldsymbol{u} \, dA_0 = 0. \tag{11}$$

We will employ a Newton-Raphson scheme to solve for the nodal displacements and hence we would need the tangent matrix. For that purpose we obtain the linearized form of the residual $\mathcal{R}$ as

$$d\mathcal{R} = \frac{\partial \mathcal{R}}{\partial \boldsymbol{u}} \cdot d\boldsymbol{u} = \int_{\Omega_0} dS_{IJ} \, \delta E_{IJ} \, dV_0 + \int_{\Omega_0} S_{IJ} \, d(\delta E_{IJ}) \, dV_0. \tag{12}$$

Defining the modulus tensor as

$$\boldsymbol{D} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} \tag{13}$$

and using (13) in (12) we get

$$d\mathcal{R} = \int_{\Omega_0} D_{IJKL} \, \delta E_{IJ} \, dE_{KL} \, dV_0 + \int_{\Omega_0} S_{IJ} \, d(\delta E_{IJ}) \, dV_0. \tag{14}$$

Finally, using (9) in (14) we get

$$d\mathcal{R} = \int_{\Omega_0} J d_{ijkl} \, \delta\epsilon_{ij} \, d\epsilon_{kl} \, dV_0 + \int_{\Omega_0} S_{IJ} \, d(\delta E_{IJ}) \, dV_0. \tag{15}$$

where

$$J d_{ijkl} = F_{iI} F_{jJ} F_{kK} F_{lL} D_{IJKL} \tag{16}$$

is the modulus tensor in the deformed configuration. Now let us simplify the last integrand of (15). Noticing that $\delta \boldsymbol{E}$ given by (9) can alternatively be expressed as $\delta \boldsymbol{E} = \text{sym}(\boldsymbol{F}^T \nabla_0 \delta \boldsymbol{u})$ we obtain

$$\begin{aligned} d(\delta \boldsymbol{E}) &= \frac{1}{2}\left(\nabla_0 d\boldsymbol{u}^T \nabla_0 \delta \boldsymbol{u} + \nabla_0 \delta \boldsymbol{u}^T \nabla_0 d\boldsymbol{u}\right), \\ &= \frac{1}{2}\left(\boldsymbol{F}^T \nabla d\boldsymbol{u}^T \nabla \delta \boldsymbol{u} \, \boldsymbol{F} + \boldsymbol{F}^T \nabla \delta \boldsymbol{u}^T \nabla d\boldsymbol{u} \, \boldsymbol{F}\right) \end{aligned} \tag{17}$$

where sym(·) denotes the symmetric part of the tensor in the argument. In deriving (17) we have used the fact that the virtual displacement $\delta\boldsymbol{u}$ is independent of the displacement field $\boldsymbol{u}$. Using (17) along with the standard relation $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T}$ we can rewrite the second integrand of (15) as

$$\boldsymbol{S} : d(\delta\boldsymbol{E}) = \nabla d\boldsymbol{u}\,\boldsymbol{\tau} : \nabla\delta\boldsymbol{u}. \tag{18}$$

Using (18) we finally express the linearization term as

$$d\mathcal{R} = \int_{\Omega_0} J d_{ijkl}\,\delta\epsilon_{ij}\,d\epsilon_{kl}\,dV_0 + \int_{\Omega_0} (\nabla d\boldsymbol{u})_{ik}\tau_{kj}(\nabla\delta\boldsymbol{u})_{ij}\,dV_0. \tag{19}$$

References

[1] O. C. Zienkiewicz and R. L. Taylor. *The Finite Element Method: Volume 2- Solid Mechanics.* Butterworth-Heinemann, Oxford, 2000.