Additional derivatives of equality and inequality constraints w.r.t Polar and Cartesian coordinates using Complex Matrix Notation

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Contents

Notation

N_b	: number of buses in the network
N_g	: number of generator buses
$V_k = V_k^r + \imath V_k^m$	: complex voltage at bus k using Cartesian coordinates
V_k^r, V_k^m	: real and imaginary part of complex voltage at bus k
$V_k = \lvert V_k \rvert e^{\imath \delta_k}$	: complex voltage at bus k using Polar coordinates
$\lvert V_k \rvert, \delta_k$	: voltage magnitude and angle at bus k
P_k^g, Q_k^g	: active and reactive power generations at bus k

1 Introduction

In this document, the additional first and second-order partial derivatives of the power balance and flow equations w.r.t Polar and Cartesian coordinates is computed using complex matrix notation as shown in Matpower.

2 Derivatives w.r.t $\mathbf{x} = [\vec{V}^m, \vec{V}^r, \vec{P}^g, \vec{Q}^g]$

2.1 Bus Voltages

2.1.1 First derivatives

$$\vec{V}_{\vec{V}^m} = \frac{\partial \vec{V}}{\partial \vec{V}^m} = \imath[\vec{1}] \tag{1}$$

$$\vec{V}_{\vec{V}^r} = \frac{\partial \vec{V}}{\partial \vec{V}^r} = [\vec{1}] \tag{2}$$

$$\vec{V}^{-1}_{\vec{V}^m} = \frac{\partial}{\partial \vec{V}^m}\left(\frac{1}{\vec{V}}\right) = -\imath[\vec{V}]^{-2} \tag{3}$$

$$\vec{V}^{-1}_{\vec{V}^r} = \frac{\partial}{\partial \vec{V}^r}\left(\frac{1}{\vec{V}}\right) = -[\vec{V}]^{-2} \tag{4}$$

2.2 Bus Complex Current Injections

$$\vec{I}_{bus} = Y_{bus}\vec{V} \tag{5}$$

2.2.1 First derivatives

$$\frac{\partial \vec{I}_{bus}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \vec{I}_{bus}}{\partial \vec{V}^m} & \frac{\partial \vec{I}_{bus}}{\partial \vec{V}^r} & 0 & 0 \end{bmatrix} \tag{6}$$

$$\frac{\partial \vec{I}_{bus}}{\partial \vec{V}^m} = Y_{bus}\frac{\partial \vec{V}}{\partial \vec{V}^m} = \imath Y_{bus} \tag{7}$$

$$\frac{\partial \vec{I}_{bus}}{\partial \vec{V}^r} = Y_{bus}\frac{\partial \vec{V}}{\partial \vec{V}^r} = Y_{bus} \tag{8}$$

2.3 Current-mismatch

$$\begin{aligned} G(\mathbf{x}) &= \Delta\vec{I}_{bus} = \vec{I}_{bus} - \vec{I}^{sp} \\ &= Y_{bus}\vec{V} - \left([\vec{S}^{sp}]\vec{V}^{-1}\right)^* \qquad (N_b \times 1) \end{aligned} \tag{9}$$

2.3.1 First Derivatives

$$G_{\mathbf{x}} = \frac{\partial G}{\partial \mathbf{x}} = [G_{\vec{V}^m} \quad G_{\vec{V}^r} \quad G_{\vec{P}^g} \quad G_{\vec{Q}^g}] \quad (N_b \times 2(N_b + N_g)) \tag{10}$$

$$\begin{aligned} G_{\vec{V}^m} = \frac{\partial G}{\partial \vec{V}^m} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{V}^m} - [\vec{S}^{sp^*}]\frac{\partial\left(\vec{V}^{-1}\right)^*}{\partial \vec{V}^m} \\ &= \imath\left(Y_{bus} - [\vec{S}^{sp^*}][\vec{V}^*]^{-2}\right) \qquad (N_b \times N_b) \end{aligned} \tag{11}$$

$$\begin{aligned} G_{\vec{V}^r} = \frac{\partial G}{\partial \vec{V}^r} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{V}^r} - [\vec{S}^{sp^*}]\frac{\partial\left(\vec{V}^{-1}\right)^*}{\partial \vec{V}^r} \\ &= Y_{bus} + [\vec{S}^{sp^*}][\vec{V}^*]^{-2} \end{aligned} \tag{12}$$

$$\begin{aligned} G_{\vec{P}^g} = \frac{\partial G}{\partial \vec{P}^g} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{P}^g} - [\vec{V}^*]^{-1}\frac{\partial \vec{S}^{sp^*}}{\partial \vec{P}^g} \\ &= -[\vec{V}^*]^{-1}C_g \qquad (N_b \times N_g) \end{aligned} \tag{13}$$

$$\begin{aligned} G_{\vec{Q}^g} = \frac{\partial G}{\partial \vec{Q}^g} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{Q}^g} - [\vec{V}^*]^{-1}\frac{\partial \vec{S}^{sp^*}}{\partial \vec{Q}^g} \\ &= \imath[\vec{V}^*]^{-1}C_g \qquad (N_b \times N_g) \end{aligned} \tag{14}$$

where C_g is $N_b \times N_g$ generator connection matrix.

2.3.2 Second Derivatives

$$\begin{aligned} G_{\mathbf{xx}}(\vec{\lambda}) &= \frac{\partial}{\partial \mathbf{x}}(G_{\mathbf{x}}^T\vec{\lambda}) \in \mathbb{C}^{2(N_b+N_g)\times 2(N_b+N_g)} \\ &= \begin{bmatrix} G_{\vec{V}^m\vec{V}^m}(\vec{\lambda}) & G_{\vec{V}^m\vec{V}^r}(\vec{\lambda}) & G_{\vec{V}^m\vec{P}^g}(\vec{\lambda}) & G_{\vec{V}^m\vec{Q}^g}(\vec{\lambda}) \\ G_{\vec{V}^r\vec{V}^m}(\vec{\lambda}) & G_{\vec{V}^r\vec{V}^r}(\vec{\lambda}) & G_{\vec{V}^r\vec{P}^g}(\vec{\lambda}) & G_{\vec{V}^r\vec{Q}^g}(\vec{\lambda}) \\ G_{\vec{P}^g\vec{V}^m}(\vec{\lambda}) & G_{\vec{P}^g\vec{V}^r}(\vec{\lambda}) & 0 & 0 \\ G_{\vec{Q}^g\vec{V}^m}(\vec{\lambda}) & G_{\vec{Q}^g\vec{V}^r}(\vec{\lambda}) & 0 & 0 \end{bmatrix} \end{aligned} \tag{15}$$

$$
\begin{aligned}
G_{\vec{P}^g\vec{V}^m}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^m}(G_{\vec{P}^g}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^m}\big(-C_g^T[\vec{V}^*]^{-1}\vec{\lambda}\big) \\
&= -\imath C_g^T[\vec{\lambda}][\vec{V}^*]^{-2} \qquad (N_g \times N_b) \qquad (16) \\
G_{\vec{P}^g\vec{V}^r}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^r}(G_{\vec{P}^g}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^r}\big(-C_g^T[\vec{V}^*]^{-1}\vec{\lambda}\big) \\
&= C_g^T[\vec{\lambda}][\vec{V}^*]^{-2} \qquad (N_g \times N_b) \qquad (17) \\
G_{\vec{Q}^g\vec{V}^m}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^m}(G_{\vec{Q}^g}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^m}(\imath C_g^T[\vec{V}^*]^{-1}\vec{\lambda}) \\
&= -C_g^T[\vec{\lambda}][\vec{V}^*]^{-2} \qquad (N_g \times N_b) \qquad (18) \\
G_{\vec{Q}^g\vec{V}^r}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^r}(G_{\vec{Q}^g}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^r}(\imath C_g^T[\vec{V}^*]^{-1}\vec{\lambda}) \\
&= -\imath C_g^T[\vec{\lambda}][\vec{V}^*]^{-2} \qquad (N_g \times N_b) \qquad (19) \\
G_{\vec{V}^m\vec{P}^g}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{P}^g}(G_{\vec{V}^m}^T\vec{\lambda}) \\
&= -\imath\big[[\vec{V}^*]^{-2}\vec{\lambda}\big]\frac{\partial \vec{S}^{sp^*}}{\partial \vec{P}^g} \\
&= -\imath\big[[\vec{V}^*]^{-2}\vec{\lambda}\big]C_g = G_{\vec{P}^g\vec{\delta}}^T(\vec{\lambda}) \qquad (N_b \times N_g) \qquad (20) \\
G_{\vec{V}^m\vec{Q}^g}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{Q}^g}(G_{\vec{V}^m}^T\vec{\lambda}) \\
&= -\imath\big[[\vec{V}^*]^{-2}\vec{\lambda}\big]\frac{\partial \vec{S}^{sp^*}}{\partial \vec{Q}^g} \\
&= -\big[[\vec{V}^*]^{-2}\vec{\lambda}\big]C_g = G_{\vec{Q}^g\vec{\delta}}^T(\vec{\lambda}) \qquad (N_b \times N_g) \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
G_{\vec{V}^r\vec{P}^g}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{P}^g}(G^T_{\vec{V}^r}\vec{\lambda}) \\
&= \big[[\vec{V}^*]^{-2}\vec{\lambda}\big]\frac{\partial \vec{S}^{sp^*}}{\partial \vec{P}^g} \\
&= \big[[\vec{V}^*]^{-2}\vec{\lambda}\big]C_g = G^T_{\vec{P}^g\vec{V}^r}(\vec{\lambda}) \qquad (N_b \times N_g) && (22) \\
G_{\vec{V}^r\vec{Q}^g}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{Q}^g}(G^T_{\vec{V}^r}\vec{\lambda}) \\
&= \big[[|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\vec{\lambda}\big]\frac{\partial \vec{S}^{sp^*}}{\partial \vec{Q}^g} \\
&= -\imath\big[[\vec{V}^*]^{-2}\vec{\lambda}\big]C_g = G^T_{\vec{Q}^g|\vec{V}|}(\vec{\lambda}) \qquad (N_b \times N_g) && (23) \\
G_{\vec{V}^m\vec{V}^m}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^m}(G^T_{\vec{V}^m}\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^m}\Big(\imath\big(Y^T_{bus} - [\vec{S}^{sp^*}][\vec{V}^*]^{-2}\big)\vec{\lambda}\Big) \\
&= -\imath[\vec{S}^{sp^*}][\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^2}\big)^*}{\partial \vec{V}^m} \\
&= 2[\vec{S}^{sp^*}][\vec{\lambda}][\vec{V}^*]^{-3} \qquad (N_b \times N_b) && (24) \\
G_{\vec{V}^m\vec{V}^r}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^r}(G^T_{\vec{V}^m}\vec{\lambda}) \\
&= -\imath[\vec{S}^{sp^*}][\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^2}\big)^*}{\partial \vec{V}^m} \\
&= 2\imath[\vec{S}^{sp^*}][\vec{\lambda}][\vec{V}^*]^{-3} \qquad (N_b \times N_b) && (25) \\
G_{\vec{V}^r\vec{V}^r}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^r}(G^T_{\vec{V}^r}\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^r}\Big(Y^T_{bus}\vec{\lambda} + [\vec{S}^{sp^*}][\vec{V}^*]^{-2}\vec{\lambda}\Big) \\
&= [\vec{S}^{sp^*}][\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^2}\big)^*}{\partial \vec{V}^r} \\
&= -2[\vec{S}^{sp^*}][\vec{\lambda}][\vec{V}^*]^{-3} \qquad (N_b \times N_b) && (26) \\
G_{\vec{V}^r\vec{V}^m}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^m}(G^T_{\vec{V}^r}\vec{\lambda}) \\
&= \vec{S}^{sp^*}][\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^2}\big)^*}{\partial \vec{V}^m} \\
&= 2\imath[\vec{S}^{sp^*}][\vec{\lambda}][\vec{V}^*]^{-3} \qquad (N_b \times N_b) && (27)
\end{aligned}
$$

2.4 Bus Complex Power Injections

$$
\vec{S}_{bus} = [\vec{V}]\vec{I}^*_{bus} \tag{28}
$$

2.5 Power-mismatch

$$
\begin{aligned}
G(\mathbf{x}) = \Delta\vec{S} &= \vec{S}_{bus} - \vec{S}^{sp} \\
&= [\vec{V}]\vec{I}^{*}_{bus} - \vec{S}^{sp} \qquad (N_b \times 1)
\end{aligned} \tag{29}
$$

2.5.1 First derivatives

$$
G_{\mathbf{x}} = \frac{\partial G}{\partial \mathbf{x}} = [G_{\vec{V}^m} \quad G_{\vec{V}^r} \quad G_{\vec{P}^g} \quad G_{\vec{Q}^g}] \quad (N_b \times 2(N_b + N_g)) \tag{30}
$$

$$
\begin{aligned}
G_{\vec{V}^m} = \frac{\partial \vec{S}_{bus}}{\partial \vec{V}^m} &= [\vec{I}^{*}_{bus}]\frac{\partial \vec{V}}{\partial \vec{V}^m} + [\vec{V}]\frac{\partial \vec{I}^{*}_{bus}}{\partial \vec{V}^m} \\
&= \imath\Big([\vec{I}^{*}_{bus}] - [\vec{V}]Y^{*}_{bus}\Big) \qquad (N_b \times N_b)
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
G_{\vec{V}^r} = \frac{\partial \vec{S}_{bus}}{\partial \vec{V}^r} &= [\vec{I}^{*}_{bus}]\frac{\partial \vec{V}}{\partial \vec{V}^r} + [\vec{V}]\frac{\partial \vec{I}^{*}_{bus}}{\partial \vec{V}^r} \\
&= [\vec{I}^{*}_{bus}] + [\vec{V}]Y^{*}_{bus} \qquad (N_b \times N_b)
\end{aligned} \tag{32}
$$

$$
G_{\vec{P}^g} = -\frac{\partial \vec{S}^{sp}}{\partial \vec{P}^g} = -C_g \qquad (N_b \times N_g) \tag{33}
$$

$$
G_{\vec{Q}^g} = -\frac{\partial \vec{S}^{sp}}{\partial \vec{Q}^g} = -\imath C_g \qquad (N_b \times N_g) \tag{34}
$$

2.5.2 Second derivatives

$$
\begin{aligned}
G_{\mathbf{xx}}(\vec{\lambda}) &= \frac{\partial}{\partial \mathbf{x}}(G^T_{\mathbf{x}}\vec{\lambda}) \in \mathbb{C}^{2(N_b+N_g)\times 2(N_b+N_g)} \\
&= \begin{bmatrix} G_{\vec{V}^m\vec{V}^m}(\vec{\lambda}) & G_{\vec{V}^m\vec{V}^r}(\vec{\lambda}) & 0 & 0 \\ G_{\vec{V}^r\vec{V}^m}(\vec{\lambda}) & G_{\vec{V}^r\vec{V}^r}(\vec{\lambda}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
G_{\vec{V}^m\vec{V}^m}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^m}(G_{\vec{V}^m}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^m}\Big(\imath([\vec{I}_{bus}^*] - {Y_{bus}^*}^T[\vec{V}])\vec{\lambda}\Big) \\
&= \imath\Big([\vec{\lambda}]\frac{\partial \vec{I}_{bus}^*}{\partial \vec{V}^m} - {Y_{bus}^*}^T[\vec{\lambda}]\frac{\partial \vec{V}}{\partial \vec{V}^m}\Big) \\
&= \imath\Big(-\imath[\vec{\lambda}]Y_{bus}^* - \imath{Y_{bus}^*}^T[\vec{\lambda}][\vec{1}]\Big) \\
&= [\vec{\lambda}]Y_{bus}^* + {Y_{bus}^*}^T[\vec{\lambda}] \qquad (N_b \times N_b) && (36) \\
G_{\vec{V}^m\vec{V}^r}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^r}(G_{\vec{V}^m}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^r}\Big(\imath([\vec{I}_{bus}^*] - {Y_{bus}^*}^T[\vec{V}])\vec{\lambda}\Big) \\
&= \imath\Big([\vec{\lambda}]\frac{\partial \vec{I}_{bus}^*}{\partial \vec{V}^r} - {Y_{bus}^*}^T[\vec{\lambda}]\frac{\partial \vec{V}}{\partial \vec{V}^r}\Big) \\
&= \imath\Big([\vec{\lambda}]Y_{bus}^* - {Y_{bus}^*}^T[\vec{\lambda}]\Big) \qquad (N_b \times N_b) && (37) \\
G_{\vec{V}^r\vec{V}^r}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^r}(G_{\vec{V}^r}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^r}\Big(([\vec{I}_{bus}^*] + {Y_{bus}^*}^T[\vec{V}])\vec{\lambda}\Big) \\
&= [\vec{\lambda}]\frac{\partial \vec{I}_{bus}^*}{\partial \vec{V}^r} + {Y_{bus}^*}^T[\vec{\lambda}]\frac{\partial \vec{V}}{\partial \vec{V}^r} \\
&= [\vec{\lambda}]Y_{bus}^* + {Y_{bus}^*}^T[\vec{\lambda}] \qquad (N_b \times N_b) && (38) \\
G_{\vec{V}^r\vec{V}^m}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{V}^m}(G_{\vec{V}^r}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial \vec{V}^m}\Big(([\vec{I}_{bus}^*] + {Y_{bus}^*}^T[\vec{V}])\vec{\lambda}\Big) \\
&= [\vec{\lambda}]\frac{\partial \vec{I}_{bus}^*}{\partial \vec{V}^m} + {Y_{bus}^*}^T[\vec{\lambda}]\frac{\partial \vec{V}}{\partial \vec{V}^m} \\
&= -\imath\Big([\vec{\lambda}]Y_{bus}^* - {Y_{bus}^*}^T[\vec{\lambda}]\Big) \qquad (N_b \times N_b) && (39)
\end{aligned}
$$

2.6 Branch Voltages

$$\vec{V}_f = C_f V \tag{40}$$

$$\vec{V}_t = C_t V \tag{41}$$

2.6.1 First derivatives

$$\frac{\partial \vec{V}_f}{\partial \vec{V}^m} = \imath C_f[\vec{1}] = \imath C_f \tag{42}$$

$$\frac{\partial \vec{V}_f}{\partial \vec{V}^r} = C_f[\vec{1}] = C_f. \tag{43}$$

2.7 Branch Complex Currents

$$\vec{I}_f = Y_f \vec{V} \tag{44}$$
$$\vec{I}_t = Y_t \vec{V} \tag{45}$$

2.7.1 First derivatives

$$\frac{\partial \vec{I}_f}{\partial \mathbf{x}} = \left[\frac{\partial \vec{I}_f}{\partial \vec{V}^m} \quad \frac{\partial \vec{I}_f}{\partial \vec{V}^r} \quad 0 \quad 0 \right] \tag{46}$$
$$\frac{\partial \vec{I}_f}{\partial \vec{V}^m} = Y_f \frac{\partial \vec{V}}{\partial \vec{V}^m} = \imath Y_f \tag{47}$$
$$\frac{\partial \vec{I}_f}{\partial \vec{V}^r} = Y_f \frac{\partial \vec{V}}{\partial \vec{V}^r} = Y_f \tag{48}$$

2.8 Branch Complex Power Flows

$$\vec{S}^f = [\vec{V}_f] \vec{I}_f^* \tag{49}$$
$$\vec{S}^t = [\vec{V}_t] \vec{I}_f^* \tag{50}$$

2.8.1 First derivatives

$$\vec{S}^f_{\mathbf{x}} = \frac{\partial \vec{S}^f}{\partial \mathbf{x}} = [\vec{S}^f_{\vec{V}^m} \quad \vec{S}^f_{\vec{V}^r} \quad 0 \quad 0] \tag{51}$$

$$\begin{aligned} \vec{S}^f_{\vec{V}^m} = \frac{\partial \vec{S}^f}{\partial \vec{V}^m} &= [\vec{I}_f^*] \frac{\partial \vec{V}_f}{\partial \vec{V}^m} + [\vec{V}_f] \frac{\partial \vec{I}_f^*}{\partial \vec{V}^m} \\ &= \imath \Big([\vec{I}_f^*] C_f - [\vec{V}_f] Y_f^* \Big) \qquad (N_b \times N_b) \end{aligned} \tag{52}$$

$$\begin{aligned} \vec{S}^f_{\vec{V}^r} = \frac{\partial \vec{S}^f}{\partial \vec{V}^r} &= [\vec{I}_f^*] \frac{\partial \vec{V}_f}{\partial \vec{V}^r} + [\vec{V}_f] \frac{\partial \vec{I}_f^*}{\partial \vec{V}^r} \\ &= [\vec{I}_f^*] C_f + [\vec{V}_f] Y_f^* \qquad (N_b \times N_b) \end{aligned} \tag{53}$$

2.8.2 Second derivatives

$$\begin{aligned} \vec{S}^f_{\mathbf{xx}}(\vec{\mu}) &= \frac{\partial}{\partial \mathbf{x}} ((\vec{S}^f_{\mathbf{x}})^T \vec{\mu}) \\ &= \begin{bmatrix} \vec{S}^f_{\vec{V}^m \vec{V}^m}(\vec{\mu}) & \vec{S}^f_{\vec{V}^m \vec{V}^r}(\vec{\mu}) & 0 & 0 \\ \vec{S}^f_{\vec{V}^r \vec{V}^m}(\vec{\mu}) & \vec{S}^f_{\vec{V}^r \vec{V}^r}(\vec{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{54}$$

$$
\begin{aligned}
\vec{S}^f_{\vec{V}^m\vec{V}^m}(\vec{\mu}) &= \frac{\partial}{\partial \vec{V}^m}\Big((\vec{S}^f_{\vec{V}^m})^T\vec{\mu}\Big) \\
&= \frac{\partial}{\partial \vec{V}^m}\Big(\imath\big(C_f^T[\vec{I}_f^*] - {Y_f^*}^T[\vec{V}_f]\big)\vec{\mu}\Big) \\
&= \imath\Big(C_f^T[\vec{\mu}]\frac{\partial \vec{I}_f^*}{\partial \vec{V}^m} - {Y_f^*}^T[\vec{\mu}]\frac{\partial \vec{V}_f}{\partial \vec{V}^m}\Big) \\
&= C_f^T[\vec{\mu}]Y_f^* + {Y_f^*}^T[\vec{\mu}]C_f && (55) \\
\vec{S}^f_{\vec{V}^m\vec{V}^r}(\vec{\mu}) &= \frac{\partial}{\partial \vec{V}^r}\Big((\vec{S}^f_{\vec{V}^m})^T\vec{\mu}\Big) \\
&= \imath\Big(C_f^T[\vec{\mu}]\frac{\partial \vec{I}_f^*}{\partial \vec{V}^r} - {Y_f^*}^T[\vec{\mu}]\frac{\partial \vec{V}_f}{\partial \vec{V}^r}\Big) \\
&= \imath\Big(C_f^T[\vec{\mu}]Y_f^* - {Y_f^*}^T[\vec{\mu}]C_f\Big) && (56) \\
\vec{S}^f_{\vec{V}^r\vec{V}^r}(\vec{\mu}) &= \frac{\partial}{\partial \vec{V}^r}\Big((\vec{S}^f_{\vec{V}^r})^T\vec{\mu}\Big) \\
&= \frac{\partial}{\partial \vec{V}^r}\Big(C_f^T[\vec{I}_f^*]\vec{\mu} + {Y_f^*}^T[\vec{V}_f]\vec{\mu}\Big) \\
&= C_f^T[\vec{\mu}]\frac{\partial \vec{I}_f^*}{\partial \vec{V}^r} + {Y_f^*}^T[\vec{\mu}]\frac{\partial \vec{V}_f}{\partial \vec{V}^r} \\
&= C_f^T[\vec{\mu}]Y_f^* + {Y_f^*}^T[\vec{\mu}]C_f && (57) \\
\vec{S}^f_{\vec{V}^r\vec{V}^m}(\vec{\mu}) &= \frac{\partial}{\partial \vec{V}^m}\Big((\vec{S}^f_{\vec{V}^r})^T\vec{\mu}\Big) \\
&= C_f^T[\vec{\mu}]\frac{\partial \vec{I}_f^*}{\partial \vec{V}^m} + {Y_f^*}^T[\vec{\mu}]\frac{\partial \vec{V}_f}{\partial \vec{V}^m} \\
&= -\imath\Big(C_f^T[\vec{\mu}]Y_f^* - {Y_f^*}^T[\vec{\mu}]C_f\Big) && (58)
\end{aligned}
$$

3 Derivatives w.r.t $\mathbf{x} = [\vec{\delta}, |\vec{V}|, \vec{P}^g, \vec{Q}^g]$

3.1 Current-mismatch

$$
\begin{aligned}
G(\mathbf{x}) = \Delta\vec{I}_{bus} &= \vec{I}_{bus} - \vec{I}^{sp} \\
&= Y_{bus}\vec{V} - \Big([\vec{S}^{sp}]\vec{V}^{-1}\Big)^* \qquad (N_b \times 1) && (59)
\end{aligned}
$$

$$
G_{\mathbf{x}} = \frac{\partial G}{\partial \mathbf{x}} = [G_{\vec{\delta}} \quad G_{|\vec{V}|} \quad G_{\vec{P}^g} \quad G_{\vec{Q}^g}] \quad (N_b \times 2(N_b + N_g)) \qquad (60)
$$

3.1.1 First Derivatives

$$
\begin{aligned}
G_{\vec{\delta}} = \frac{\partial G}{\partial \vec{\delta}} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{\delta}} - [\vec{S}^{sp^*}]\frac{\partial \left(\vec{V}^{-1}\right)^*}{\partial \vec{\delta}} \\
&= \imath\left(Y_{bus}[\vec{V}] - [\vec{S}^{sp^*}][\vec{V}^*]^{-1}\right) \qquad (N_b \times N_b) && (61) \\
G_{|\vec{V}|} = \frac{\partial G}{\partial |\vec{V}|} &= Y_{bus}\frac{\partial \vec{V}}{\partial |\vec{V}|} - [\vec{S}^{sp^*}]\frac{\partial \left(\vec{V}^{-1}\right)^*}{\partial |\vec{V}|} \\
&= Y_{bus}[\vec{V}][|\vec{V}|]^{-1} + [\vec{S}^{sp^*}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1} \\
&= Y_{bus}[\vec{E}] + [\vec{S}^{sp^*}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1} \qquad (N_b \times N_b) && (62) \\
G_{\vec{P}^g} = \frac{\partial G}{\partial \vec{P}^g} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{P}^g} - [\vec{V}^*]^{-1}\frac{\partial \vec{S}^{sp^*}}{\partial \vec{P}^g} \\
&= -[\vec{V}^*]^{-1}C_g \qquad (N_b \times N_g) && (63) \\
G_{\vec{Q}^g} = \frac{\partial G}{\partial \vec{Q}^g} &= Y_{bus}\frac{\partial \vec{V}}{\partial \vec{Q}^g} - [\vec{V}^*]^{-1}\frac{\partial \vec{S}^{sp^*}}{\partial \vec{Q}^g} \\
&= \imath[\vec{V}^*]^{-1}C_g \qquad (N_b \times N_g) && (64)
\end{aligned}
$$

3.1.2 Second Derivatives

$$
\begin{aligned}
G_{\mathbf{xx}}(\vec{\lambda}) &= \frac{\partial}{\partial \vec{x}}(G_{\mathbf{x}}^T\vec{\lambda}) \in \mathbb{C}^{2(N_b+N_g)\times 2(N_b+N_g)} \\
&= \begin{bmatrix}
G_{\vec{\delta}\vec{\delta}}(\vec{\lambda}) & G_{\vec{\delta}|\vec{V}|}(\vec{\lambda}) & G_{\vec{\delta}\vec{P}^g}(\vec{\lambda}) & G_{\vec{\delta}\vec{Q}^g}(\vec{\lambda}) \\
G_{|\vec{V}|\vec{\delta}}(\vec{\lambda}) & G_{|\vec{V}||\vec{V}|}(\vec{\lambda}) & G_{|\vec{V}|\vec{P}^g}(\vec{\lambda}) & G_{|\vec{V}|\vec{Q}^g}(\vec{\lambda}) \\
G_{\vec{P}^g\vec{\delta}}(\vec{\lambda}) & G_{\vec{P}^g|\vec{V}|}(\vec{\lambda}) & 0 & 0 \\
G_{\vec{Q}^g\vec{\delta}}(\vec{\lambda}) & G_{\vec{Q}^g|\vec{V}|}(\vec{\lambda}) & 0 & 0
\end{bmatrix} && (65)
\end{aligned}
$$

$$
\begin{aligned}
G_{\vec{\delta}\vec{\delta}}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{\delta}}(G_{\vec{\delta}}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial\vec{\delta}}\Big(\imath\big([\vec{V}]Y_{bus}^T - [\vec{S}^{sp^*}][\vec{V}^*]^{-1}\big)\vec{\lambda}\Big) \\
&= \imath\Big(\big[Y_{bus}^T\vec{\lambda}\big]\frac{\partial\vec{V}}{\partial\vec{\delta}} - [\vec{S}^{sp^*}][\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^*}\big)}{\partial\vec{\delta}}\Big) \\
&= -\big[Y_{bus}^T\vec{\lambda}\big][\vec{V}] + [\vec{S}^{sp^*}][\vec{\lambda}][\vec{V}^*]^{-1} \qquad (N_b \times N_b) \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
G_{\vec{\delta}|\vec{V}|}(\vec{\lambda}) &= \frac{\partial}{\partial|\vec{V}|}(G_{\vec{\delta}}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial|\vec{V}|}\Big(\imath\big([\vec{V}]Y_{bus}^T - [\vec{S}^{sp^*}][\vec{V}^*]^{-1}\big)\vec{\lambda}\Big) \\
&= \imath\Big(\big[Y_{bus}^T\vec{\lambda}\big]\frac{\partial\vec{V}}{\partial|\vec{V}|} - [\vec{S}^{sp^*}][\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^*}\big)}{\partial|\vec{V}|}\Big) \\
&= \imath\Big(\big[Y_{bus}^T\vec{\lambda}\big][\vec{V}][|\vec{V}|]^{-1} + [\vec{S}^{sp^*}][\vec{\lambda}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\Big) \qquad (N_b \times N_b) \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
G_{|\vec{V}|\vec{\delta}}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{\delta}}(G_{|\vec{V}|}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial\vec{\delta}}\Big(\big([\vec{E}]Y_{bus}^T + [\vec{S}^{sp^*}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\big)\vec{\lambda}\Big) \\
&= \big[Y_{bus}^T\vec{\lambda}\big]\frac{\partial\vec{E}}{\partial\vec{\delta}} + [\vec{S}^{sp^*}][|\vec{V}|]^{-1}[\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^*}\big)}{\partial\vec{\delta}} \\
&= \imath\Big(\big[Y_{bus}^T\vec{\lambda}\big][\vec{E}] + [\vec{S}^{sp^*}][|\vec{V}|]^{-1}[\vec{\lambda}][\vec{V}^*]^{-1}\Big) \qquad (N_b \times N_b) \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
G_{|\vec{V}||\vec{V}|}(\vec{\lambda}) &= \frac{\partial}{\partial|\vec{V}|}(G_{|\vec{V}|}^T\vec{\lambda}) \\
&= \frac{\partial}{\partial|\vec{V}|}\Big(\big([\vec{E}]Y_{bus}^T + [\vec{S}^{sp^*}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\big)\vec{\lambda}\Big) \\
&= \big[Y_{bus}^T\vec{\lambda}\big]\frac{\partial\vec{E}}{\partial|\vec{V}|} + [\vec{S}^{sp^*}]\Big(\big[[\vec{V}^*]^{-1}\vec{\lambda}\big]\frac{\partial\big(\frac{1}{|\vec{V}|}\big)}{\partial|\vec{V}|} + [|\vec{V}|]^{-1}[\vec{\lambda}]\frac{\partial\big(\frac{1}{\vec{V}^*}\big)}{\partial|\vec{V}|}\Big) \\
&= -[\vec{S}^{sp^*}]\Big(\big[[\vec{V}^*]^{-1}\vec{\lambda}\big][|\vec{V}|]^{-2} + [|\vec{V}|]^{-1}[\vec{\lambda}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\Big) \\
&= -2[\vec{S}^{sp^*}][\vec{V}^*]^{-1}[\vec{\lambda}][|\vec{V}|]^{-2} \qquad (N_b \times N_b) \qquad (69)
\end{aligned}
$$

$$
\begin{aligned}
G_{\vec{P}^g\vec{\delta}}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{\delta}}(G^T_{\vec{P}^g}\vec{\lambda}) \\
&= \frac{\partial}{\partial\vec{\delta}}\big(-C_g^T[\vec{V}^*]^{-1}\vec{\lambda}\big) \\
&= -\imath C_g^T[\vec{\lambda}][\vec{V}^*]^{-1} \qquad (N_g \times N_b) && (70) \\
G_{\vec{P}^g|\vec{V}|}(\vec{\lambda}) &= \frac{\partial}{\partial|\vec{V}|}(G^T_{\vec{P}^g}\vec{\lambda}) \\
&= \frac{\partial}{\partial|\vec{V}|}\big(-C_g^T[\vec{V}^*]^{-1}\vec{\lambda}\big) \\
&= C_g^T[\vec{\lambda}][|\vec{V}|]^{-1}[\vec{V}^*]^{-1} \qquad (N_g \times N_b) && (71) \\
G_{\vec{Q}^g\vec{\delta}}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{\delta}}(G^T_{\vec{Q}^g}\vec{\lambda}) \\
&= \frac{\partial}{\partial\vec{\delta}}(\imath C_g^T[\vec{V}^*]^{-1}\vec{\lambda}) \\
&= -C_g^T[\vec{\lambda}][\vec{V}^*]^{-1} \qquad (N_g \times N_b) && (72) \\
G_{\vec{Q}^g|\vec{V}|}(\vec{\lambda}) &= \frac{\partial}{\partial|\vec{V}|}(G^T_{\vec{Q}^g}\vec{\lambda}) \\
&= \frac{\partial}{\partial|\vec{V}|}(\imath C_g^T[\vec{V}^*]^{-1}\vec{\lambda}) \\
&= -\imath C_g^T[\vec{\lambda}][\vec{V}^*]^{-1}[|\vec{V}|]^{-1} \qquad (N_g \times N_b) && (73) \\
G_{\vec{\delta}\vec{P}^g}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{P}^g}(G^T_{\vec{\delta}}\vec{\lambda}) \\
&= -\imath\big[[\vec{V}^*]^{-1}\vec{\lambda}\big]\frac{\partial\vec{S}^{sp^*}}{\partial\vec{P}^g} \\
&= -\imath\big[[\vec{V}^*]^{-1}\vec{\lambda}\big]C_g = G^T_{\vec{P}^g\vec{\delta}}(\vec{\lambda}) \qquad (N_b \times N_g) && (74) \\
G_{\vec{\delta}\vec{Q}^g}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{Q}^g}(G^T_{\vec{\delta}}\vec{\lambda}) \\
&= -\imath\big[[\vec{V}^*]^{-1}\vec{\lambda}\big]\frac{\partial\vec{S}^{sp^*}}{\partial\vec{Q}^g} \\
&= -\big[[\vec{V}^*]^{-1}\vec{\lambda}\big]C_g = G^T_{\vec{Q}^g\vec{\delta}}(\vec{\lambda}) \qquad (N_b \times N_g) && (75) \\
G_{|\vec{V}|\vec{P}^g}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{P}^g}(G^T_{|\vec{V}|}\vec{\lambda}) \\
&= \big[[|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\vec{\lambda}\big]\frac{\partial\vec{S}^{sp^*}}{\partial\vec{P}^g} \\
&= \big[[|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\vec{\lambda}\big]C_g = G^T_{\vec{P}^g|\vec{V}|}(\vec{\lambda}) \qquad (N_b \times N_g) && (76) \\
G_{|\vec{V}|\vec{Q}^g}(\vec{\lambda}) &= \frac{\partial}{\partial\vec{Q}^g}(G^T_{|\vec{V}|}\vec{\lambda}) \\
&= \big[[|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\vec{\lambda}\big]\frac{\partial\vec{S}^{sp^*}}{\partial\vec{Q}^g} \\
&= -\imath\big[[|\vec{V}|]^{-1}[\vec{V}^*]^{-1}\vec{\lambda}\big]C_g = G^T_{\vec{Q}^g|\vec{V}|}(\vec{\lambda}) \qquad (N_b \times N_g) && (77)
\end{aligned}
$$